

# A new approach to the results of Kövari, Sós, and Turán concerning rectangle-free subsets of the grid

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## Abstract

For positive integers  $m$  and  $n$ , define  $f(m, n)$  to be the smallest integer such that any subset  $A$  of the  $m \times n$  integer grid with  $|A| \geq f(m, n)$  contains a rectangle; that is, there are  $x \in [m]$  and  $y \in [n]$  and  $d_1, d_2 \in \mathbb{Z}^+$  such that all four points  $(x, y)$ ,  $(x + d_1, y)$ ,  $(x, y + d_2)$ , and  $(x + d_1, y + d_2)$  are contained in  $A$ . In [12], Kövari, Sós, and Turán showed that  $\lim_{k \rightarrow \infty} \frac{f(k, k)}{k^{3/2}} = 1$ . They also showed that  $f(p^2, p^2 + p) = p^2(p + 1) + 1$  whenever  $p$  is a prime number. We recover their asymptotic result and strengthen the second, providing cleaner proofs which exploit a connection to projective planes, first noticed by Mendelsohn in [14]. We also provide an explicit lower bound for  $f(k, k)$  which holds for all  $k$ .

## 1 Introduction and motivation

For a positive integer  $n$ , let  $[n] = \{1, 2, \dots, n\}$ . For  $m, n \in \mathbb{Z}^+$ , define  $f(m, n)$  to be the least integer such that if  $A \subseteq [m] \times [n]$  with  $|A| \geq f(m, n)$ , then  $A$  contains a rectangle; that is, there is  $x \in [m]$ ,  $y \in [n]$ , and  $d_1, d_2 \in \mathbb{Z}^+$  such that all four points  $(x, y)$ ,  $(x + d_1, y)$ ,  $(x, y + d_2)$ , and  $(x + d_1, y + d_2)$  are contained in  $A$ . For ease in notation, let  $f(k) = f(k, k)$ . For  $c \in \mathbb{Z}^+$ , a  $c$ -coloring of a set  $S$  is a surjective map  $\chi : S \rightarrow [c]$ . If  $\chi$  is constant on a set  $A \subset S$ , we say that  $A$  is *monochromatic*.

We will write  $g(k) \sim h(k)$  to mean that functions  $g$  and  $h$  are *asymptotically equal*; that is,  $\lim_{k \rightarrow \infty} \frac{g(k)}{h(k)} = 1$ . Also, notice that  $f(m, n) = f(n, m)$  for any choice of  $n$  and  $m$ .

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The problem of finding bounds or exact values of  $f(m, n)$  finds its roots in the famous theorem of van der Waerden from [21], which states that given any positive integers  $c$  and  $d$ , there exists an integer  $N$  such that any  $c$ -coloring of  $[N]$  contains a monochromatic arithmetic progression of length  $d$ . Szemerédi proved a density version of this theorem in [20], using the now well-known Regularity Lemma. Progress in this area is still being made. For instance, in [3], Axenovich and the second author try to find the smallest  $k$  so that in any 2-coloring of  $[k] \times [k]$  there is a monochromatic *square*; i.e., a rectangle with  $d_1 = d_2$ . While the upper bounds are enormous, they proved  $k \geq 13$ ; in [4], Bacher and Eliahou show that  $k = 15$ . In [10], the authors are interested in finding  $\text{OBS}_c$ , which is the collection of  $[m] \times [n]$  grids which cannot be colored in  $c$  colors without a monochromatic rectangle, but every proper subgrid can be; see also [7]. For a more complete survey on van der Waerden type problems, see [11].

Zarankiewicz introduced the problem of finding  $f(m, n)$  in [22] using the language of minors of  $(0,1)$ -matrices. In [12], Kővari, Sós, and Turán show that  $f(k) \sim k^{3/2}$  and that whenever  $p$  is a prime number, we have  $f(p^2 + p, p^2) = p^2(p + 1) + 1$ . In this manuscript, we will recover this asymptotic result and strengthen the second result.

In [17], Reiman achieved the bound of

$$f(m, n) \leq \frac{1}{2} \left( m + \sqrt{m^2 + 4mn(n-1)} \right) + 1. \quad (1)$$

Notice that by setting  $m = p^2 + p$  and  $n = p^2$ , the right hand side of (1) becomes  $p^2(p + 1) + 1$ , so the result of Kővari, Sós, and Turán implies that the inequality is sharp. Reiman showed equality in (1) in the case that  $m = n = q^2 + q + 1$ , provided  $q$  is a prime power. In [14], Mendelsohn recovers and strengthens the equality result of Reiman by noticing the connection of the Zarankiewicz problem to projective planes.

A  $k \times k$   $(0,1)$ -matrix  $A$  corresponds to a subset  $S_A \subset [k] \times [k]$  by

$$(i, j) \in S \text{ if and only if the } (i, j) \text{ entry of } A \text{ is } 1.$$

Notice that the set  $S_A$  contains a rectangle if and only if the matrix  $A^T A$  has an entry off the main diagonal which is not equal to 0 or 1. Also notice that  $\text{tr}(A^T A) = |S_A|$ .

Such  $(0,1)$ -matrices arise in the study of projective planes. A projective plane of order  $n$  is an incidence structure consisting of  $n^2 + n + 1$  points and  $n^2 + n + 1$  lines such that

- (i) any two distinct points lie on exactly one line;
- (ii) any two distinct lines intersect in exactly one point;
- (iii) each line contains exactly  $n + 1$  points; and
- (iv) there is a set of 4 points such that no 3 of these points lie on the same line.

It is not known for which positive integers  $n$  there exists a projective plane of order  $n$ ; projective planes have been constructed for all prime-power orders, but for no others. In the well-known paper [5], Bruck and Ryser show that if the square-free part of  $n$  is divisible by a prime of the form  $4k + 3$ , and if  $n$  is congruent to 1 or 2 modulo 4, then there is no projective plane of order  $n$ ; see also [6]. More recently, the authors in [8] draw a connection between the existence of projective planes of order greater than or equal to 157 and the number of cycles in  $n \times n$  bipartite graphs of girth at least 6. In 1989, a computer search conducted by the authors in [13] showed that there is no projective plane of order 10. The smallest order for which it is still not known whether there is a projective plane is 12, although the results in [15, 19, 16, 1, 2] suggest that there is no such structure.

Next we state a lemma which appears in [14] connecting projective planes to the Zarankiewicz problem.

**Lemma 1.** *If  $n$  is a positive integer such that there exists a projective plane of order  $n$ , then  $f(n^2 + n + 1) = (n + 1)(n^2 + n + 1) + 1$ .*

We will include a proof of Lemma 1 both for completeness and since we will reference the lower bound construction in the proof of Theorem 2.

*Proof of Lemma 1.* Let  $n$  be a positive integer such that there is a projective plane of that order. For ease in notation, set  $N = n^2 + n + 1$ . First we will show that  $f(N) \geq (n + 1)N + 1$ .

We begin by constructing a  $N \times N$   $(0, 1)$ -matrix  $A$ . There exists a projective plane  $P$  of order  $n$ ; so let  $A$  be the  $N \times N$  matrix whose rows correspond to the points of  $P$  and whose columns correspond to the lines of  $P$  where the  $(i, j)$  entry of  $A$  is equal to 1 if and only if the point indexed by  $i$  lies on the line indexed by  $j$ . Since any two distinct lines have exactly one point in common, the scalar product of any two distinct columns must be 1; hence,  $S_A$  does not contain a rectangle. Since each line contains exactly  $(n + 1)$  points,  $|S_A| = \text{tr}(A^T A) = (n + 1)N$ , so  $f(N) \geq (n + 1)N + 1$ .

Now, suppose  $A$  is any  $N \times N$   $(0, 1)$ -matrix with  $(n + 1)N + 1$  nonzero entries, and let  $a_i$  denote the number of 1s in row  $i$ . The number of pairs of 1s in row  $i$  is  $\binom{a_i}{2}$ , so the total number of pairs of 1s from each row is  $\sum_{i=1}^N \binom{a_i}{2}$ .

The number of pairs of distinct column indices is  $\binom{N}{2}$ . If  $\sum_{i=1}^N \binom{a_i}{2} > \binom{N}{2}$ , the pigeonhole principle implies that there is a pair of column indices such that there are two distinct rows which have 1s in both of those columns; i.e.,  $S_A$  contains a rectangle.

To see that  $\sum_{i=1}^N \binom{a_i}{2} > \binom{N}{2}$ , recall that the Cauchy-Schwarz inequality

gives

$$\left(\sum_{i=1}^N a_i\right)^2 \leq \sum_{i=1}^N a_i^2 \sum_{i=1}^N 1^2. \quad (2)$$

Since  $\sum_{i=1}^N a_i = (n+1)N + 1$  by assumption, the bound in (2) gives

$$(n+1)^2 N + 2(n+1) + \frac{1}{N} \leq \sum_{i=1}^N a_i^2. \quad (3)$$

Since  $\sum_{i=1}^N a_i^2 = \sum_{i=1}^N a_i(a_i - 1) + \sum_{i=1}^N a_i = 2 \sum_{i=1}^N \binom{a_i}{2} + (n+1)N + 1$ , inequality (3) gives

$$N((n+1)^2 - (n+1)) + 2(n+1) + \frac{1}{N} - 1 \leq 2 \sum_{i=1}^N \binom{a_i}{2}. \quad (4)$$

Since  $(n+1)^2 - (n+1) = n^2 + n + 1 - 1 = N - 1$ , inequality (4) can be rewritten as

$$\frac{N(N-1)}{2} + n + \frac{1}{N} + \frac{1}{2} \leq \sum_{i=1}^N \binom{a_i}{2}, \quad (5)$$

and since  $n > 0$ , the left hand side of (5) is bound from below by  $\binom{N}{2}$ , as desired.  $\square$

It is interesting to note that we have equality in (2) just in case all of the  $a_i$  are equal; that is, each row and column contain the same number of 1s.

## 2 Main results

Our main lemma is below, a useful proposition for dealing with asymptotic behavior of functions when some explicit values of the functions are known. A version of this lemma is used in [12], but it is neither proved nor explicitly stated.

**Lemma 2.** *Suppose  $g$  and  $h$  are monotonically increasing functions. If  $a_n$  is a strictly increasing sequence of positive integers such that*

$$(i) \quad \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1;$$

$$(ii) \quad \lim_{n \rightarrow \infty} \frac{h(a_{n+1})}{h(a_n)} = 1; \text{ and}$$

$$(iii) \quad g(a_n) = h(a_n) \text{ for all } n,$$

then  $g \sim h$ .

Theorem 1 recovers the asymptotic result of Kövari, Sós, and Turán. Theorem 2 strengthens another of their results. The proofs exploit the connection to projective planes, cleaning up the arguments found in [12]. Theorem 3 is an explicit lower bound for  $f(k)$ , which holds for all  $k$ .

**Theorem 1.**  $f(k) \sim k^{3/2}$ .

**Theorem 2.** Let  $n$  be a positive integer. If there is a projective plane of order  $n$ , then  $f(n^2, n^2 + n) = n^2(n + 1) + 1$ .

**Theorem 3.** If  $k \in \mathbb{Z}$  with  $k \geq 3$ , then  $f(k) \geq \frac{1}{16}((k + 4)\sqrt{4k - 3} + 5k + 22)$ .

### 3 Proof of Lemma 2

Now we prove Lemma 2.

*Proof.* Let  $g$  and  $h$  be monotonically increasing functions. Suppose  $a_n$  is a strictly increasing sequence of positive integers such that  $\lim_{n \rightarrow \infty} \frac{h(a_{n+1})}{h(a_n)} = 1$  and that  $g(a_n) = h(a_n)$  for all  $n$ . Let  $\varepsilon > 0$ . Choose  $N$  so that

$$\left| \frac{h(a_{n+1})}{h(a_n)} - 1 \right| < \varepsilon \text{ and } \left| \frac{h(a_n)}{h(a_{n+1})} - 1 \right| < \varepsilon \quad (6)$$

whenever  $n > N$ . Next, choose  $m$  large enough so that for some  $n > N$ , we have  $a_n \leq m \leq a_{n+1}$ . Since  $g$  is increasing and  $g$  and  $h$  agree on the sequence  $a_n$ , we have

$$h(a_n) = g(a_n) \leq g(m) \leq g(a_{n+1}) = h(a_{n+1}). \quad (7)$$

Since  $h$  is monotone increasing,  $h(a_n) \leq h(m) \leq h(a_{n+1})$ , so we may transform (7) into

$$\frac{h(a_n)}{h(a_{n+1})} \leq \frac{g(m)}{h(m)} \leq \frac{h(a_{n+1})}{h(a_n)}. \quad (8)$$

Subtracting 1 from every term in (8) and taking absolute values gives that either

$$\left| \frac{g(m)}{h(m)} - 1 \right| \leq \left| \frac{h(a_{n+1})}{h(a_n)} - 1 \right| \text{ or } \left| \frac{g(m)}{h(m)} - 1 \right| \leq \left| \frac{h(a_n)}{h(a_{n+1})} - 1 \right|.$$

Without loss of generality, say  $\left| \frac{g(m)}{h(m)} - 1 \right| \leq \left| \frac{h(a_{n+1})}{h(a_n)} - 1 \right|$ . By (6), we have

$$\left| \frac{g(m)}{h(m)} - 1 \right| < \varepsilon,$$

so  $\frac{g}{h} \rightarrow 1$  and  $g \sim h$ , as desired.  $\square$

## 4 Proof of Theorem 1

Now we prove Theorem 1.

*Proof.* For a positive integer  $k$ , set

$$h(k) = \left( \sqrt{k - \frac{3}{4}} + \frac{1}{2} \right) k + 1.$$

Notice that  $h(k) \sim k^{3/2}$  and that  $h(n^2 + n + 1) = (n + 1)(n^2 + n + 1) + 1$ , so by Lemma 1, we have  $f(n^2 + n + 1) = h(n^2 + n + 1)$  whenever there is a projective plane of order  $n$ . Since there is a projective plane of order  $p$  for every prime  $p$ , we have that  $f$  and  $h$  agree on an infinite sequence of integers  $a_n$  for which  $\frac{a_{n+1}}{a_n} \rightarrow 1$  (see [18, 9]). Notice that  $\frac{h(a_{n+1})}{h(a_n)} \rightarrow 1$ , so we may apply Lemma 2 to achieve  $f \sim h$ , and thus  $f \sim k^{3/2}$ , as desired.  $\square$

## 5 Proof of Theorem 2

*Proof.* Let  $n$  be a positive integer such that there is a projective plane of order  $n$ . Set  $N = n^2 + n + 1$ . As in the proof of Lemma 1, we can construct an  $N \times N$  matrix  $A$  such that  $\text{tr}(A^T A) = (n + 1)N$  and that  $A^T A$  has only 1s off the main diagonal; hence, the corresponding subset  $S_A$  of the  $N \times N$  grid has no rectangle.

To construct an  $n^2 \times (n^2 + n)$  matrix  $B$  from  $A$ , we delete the first column of  $A$  along with all rows having a 1 in the first column. Since each row and column of  $A$  contains exactly  $n + 1$  nonzero entries, we have deleted  $n + 1$  rows and 1 column. The resulting matrix  $B$  is thus an  $n^2 \times (n^2 + n)$  matrix. Since  $A^T A$  has no entries off the main diagonal greater than 1,  $B^T B$  has no entries off the main diagonal greater than 1. Since we have deleted  $(n + 1)^2$  nonzero entries from  $A$ , we have that

$$|S_B| = (n + 1)N - (n + 1)^2 = (n + 1)(n^2 + n + 1) - (n + 1)^2 = n^2(n + 1),$$

so  $f(n^2, n^2 + n) \geq n^2(n + 1) + 1$ .

Using the inequality from Reiman (1),

$$f(n^2, n^2 + n) \leq n^2(n + 1) + 1,$$

and hence  $f(n^2, n^2 + n) = n^2(n + 1) + 1$ , as desired.  $\square$

The structure obtained by taking a projective plane and deleting a line together with all of the points on that line is called an *affine plane*. Our result is stronger than that of the authors in [12], since we need only that there is a projective plane of order  $n$ , not that  $n$  is a prime number.

## 6 Proof of Theorem 3

*Proof.* Suppose  $k$  is an integer with  $k \geq 3$ . There exists a nonnegative integer  $\alpha$  such that

$$2^{2\alpha} + 2^\alpha + 1 \leq k \leq 2^{2\alpha+2} + 2^{\alpha+1} + 1. \quad (9)$$

By focusing on the upper bound from (9), this gives  $k \leq (2^{\alpha+1} + 1/2)^2 + 3/4$ , or

$$\frac{\sqrt{k - 3/4} - 1/2}{2} \leq 2^\alpha. \quad (10)$$

Let  $g(n) = (n+1)(n^2 + n + 1) + 1$ , and let  $h(k) = \frac{\sqrt{k - 3/4} - 1/2}{2}$ . Since  $g$  is an increasing function, inequality (10) gives

$$g(h(k)) \leq g(2^\alpha). \quad (11)$$

By Lemma 1, we have  $g(n) = f(n^2 + n + 1)$  whenever there exists a projective plane of order  $n$ . Since there is a projective plane of any prime power order, (11) gives

$$g(h(k)) \leq f(2^{2\alpha} + 2^\alpha + 1). \quad (12)$$

But since  $f$  is increasing, the lower bound in (9) gives  $g(h(k)) \leq f(k)$ , and since  $g(h(k)) = \frac{1}{16}((k+4)\sqrt{4k-3} + 5k + 22)$ , we have the desired result.

We also note that while  $g(h(k)) \sim \frac{1}{8}k^{3/2}$ , which is worse than the result in Theorem 1, this lower bound holds for every choice of  $k$ , and not just those  $k$  for which there exists a projective plane of order  $k$ .  $\square$

## 7 Further Research

Trying to find the exact value of  $f(m, n)$  without conditions on  $m$  and  $n$  (that is, removing the extra hypotheses from the results in [12]) would be attractive, although this problem has been open for years, and likely requires a new idea.

The next attractive direction is to take the approach of the authors in [10], and consider colorings of rectangular grids.

Recall that  $\text{OBS}_c$  is the collection of  $[m] \times [n]$  grids which cannot be colored in  $c$  colors without a monochromatic rectangle, but every proper subgrid can be. An open problem from [10] is the *rectangle-free conjecture*: if there exists a rectangle-free subset of  $[m] \times [n]$  of size  $\lceil mn/c \rceil$ , then it is possible to color  $[m] \times [n]$  in  $c$  colors so there is no monochromatic rectangle. Since the authors in [10] have theorems which depend on the rectangle-free conjecture, resolving this conjecture either in the affirmative or the negative would result in progress for obtaining  $|\text{OBS}_c|$  or even  $\text{OBS}_c$ .

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